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On the two-parameter quantum supergroups and quantum superplanes

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Abstract. Starting from the assumption that the generators of quantum supergroup are not supercommutatives with the coordinates of quantum superplane instead of usual supercommutativity in the Manin approach, we construct in a natural way the two-parameter quantum supergroup. The virtue of this formulation is that it leads to many probable quantum superplanes associated with a given quantum supergroup. Some interesting examples are presented. As a by-product of this formulation, a correspondence between the one- and two-parameter deformed differential structures on the quantum superplane is presented.

1. Introduction

In recent years a great deal of activity has been directed towards the exploration of quantum groups and algebras [1–3]. These structures may be viewed as deformations of classical Lie groups and algebras. The latter are recovered when some parameters, called deformation parameters, take some particular values. The mathematical framework of these objects is the Hopf algebra, an algebra provided with operations called the coproduct, counit and antipode. A quantum group is a non-commutative Hopf algebra consistent with these costructures.

In the viewpoint proposed by Manin [4, 5] a quantum group is identified with endomorphisms acting upon a quantum space, called a quantum plane, the coordinates of which are non-commutative. The condition for such a mapping to be an endomorphism constitutes the quantum group commutation relations. However, this is realized with an additional requirement which consists of taking elements of the quantum group and coordinates of the quantum plane which commute with each other. The natural generalization to the supercase was also performed following the above Manin approach [5–7].

In a recent paper [8], the authors have relaxed the assumption that elements of the quantum group commute with coordinates of the quantum plane. By introducing special commutation relations depending on some q_{ij} between elements of the quantum group and non-commutative coordinates, they were naturally led to the two-parameter deformation $GL_{p,q}(2)$ of the group $GL(2)$ and its corresponding quantum planes. They showed also that there is a possibility of constructing many quantum planes for a given quantum group $GL_{p,q}(2)$.

In the present paper, we will generalize the above investigation to the supercase. Before to do this let us give the Manin approach of the quantum supergroups. The natural generalization in the case of supergroups, corresponding to the one-parameter deformation,

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is the quantum supergroup $GL_q(1|1)$ which may be viewed as the superanalogue of $GL_q(2)$. It has been studied by sevral authors [9–11]. It consists of the deformation of the supergroup of 2×2 non-singular matrices with two bosonic and two fermionic elements.

Let M be an element of $GL_q(1|1)$:

$$M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \tag{1}$$

where a, d are bosonic elements and β, γ are fermionic ones. $GL_q(1|1)$ is defined by the following supercommutation relations:

$$\begin{aligned} &a\beta = q\beta a && d\beta = q\beta d \\ &a\gamma = q\gamma a && d\gamma = q\gamma d \\ &\gamma\beta + \beta\gamma = 0 && \beta^2 = \gamma^2 = 0 \\ &ad - da = (q^{-1} - q)\beta\gamma. \end{aligned} \tag{2}$$

Manin [5] has defined, in correspondance with the quantum supergroup $GL_q(1|1)$, a quantum superplane (or quantum superspace) as a quadratic $\mathbb{Z}_2$-graded algebra generated by a pair of x bosonic and θ fermionic coordinates obeying

$$\begin{aligned} &x\theta = q\theta x \qquad q \neq 0, 1 \\ &\theta^2 = 0. \end{aligned} \tag{3}$$

$GL_q(1|1)$ appears as a supersymmetry group in the sense that the points (x', θ') and (x'', θ'') obtained from the point (x, θ) by transformation under M, equation (1), and its supertranspose

$${}^{\mathrm{st}}M = \begin{pmatrix} a & -\gamma \\ \beta & d \end{pmatrix} \tag{4}$$

respectively, satisfy the same supercommutation relations as in (3), i.e. $x'\theta' = q\theta'x'$, $\theta'^2 = 0$ and $x''\theta'' = q\theta''x''$, $\theta''^2 = 0$, where

$$M: \qquad \begin{pmatrix} x \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} x' \\ \theta' \end{pmatrix} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix} \tag{5}$$

and

$${}^{\mathrm{st}}M: \qquad \begin{pmatrix} x \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} x'' \\ \theta'' \end{pmatrix} = \begin{pmatrix} a & -\gamma \\ \beta & d \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix}. \tag{6}$$

In fact, Manin [5] did not use the supertranspose transformation to obtain half of the supercommutation relations (2), but he introduced a dual quantum superplane the basic quadratic relations of which remain invariants under transformation M.

A remarkable fact here is that relations (3) are invariant not only under the transformation M but also under its supertranspose ${}^{\mathrm{st}}M$ when one makes use of the assumption that the elements of the quantum supergroup supercommute with the coordinates of the quantum superplane. In this work we will relax the latter assumption. This naturally leads to the two-parameter deformation even though we do not suppose any restriction on the number of parameters at the outset.

Now let us recall the two-parameter quantum supergroup $GL_{p,q}(1|1)$ [5–7]. It is generated by elements of the matrix M which satisfy the supercommutation relations

$$\begin{aligned} &a\beta = p\beta a && d\beta = p\beta d \\ &a\gamma = q\gamma a && d\gamma = q\gamma d \\ &p\beta\gamma + q\gamma\beta = 0 && \beta^2 = \gamma^2 = 0 \\ &ad - da = (q^{-1} - p)\beta\gamma. \end{aligned} \tag{7}$$

It is clear that the one-parameter case in (2) is recovered in the limit $p = q$. Note that these relations (7) can be written in the form of a *super-RTT* equation [12]. It can easily be seen that the quadratic relations (3) are now not invariant under the transformation (6), but only under that in (5). However, if one requires invariance under the two tranformations with the assumption that generators of the quantum supergroup supercommute with the coordinates of the quantum superplane, one is led to the one-parameter quantum supergroup. Now, if we relax the above assumption, we will be led naturally to the two-parameter quantum supergroup [5–7] in such manner that the supercommutation relations (7) come directly from the condition that the quadratic relations (3) are preserved under both transformations (5) and (6). This will be done without taking any restriction on the number of parameters at the outset. Our starting point is some special supercommutaion relations between The elements of the quantum matrix M, equation (1), and the coordinates of the quantum superplane. It appears that the quantum supergroup generators do not supercommute with the coordinates of the quantum superplane generically.

The paper is organized as follows. In section 2, we shall construct the two-parameter quantum supergroup the elements of which do not supercommute with the coordinates of quantum superplane. This formulation leads us to many probable quantum superplanes associated with a given quantum supergroup. Section 3 is devoted to the discussion of some special examples. We elaborate also a correspondence between the one and the two-parameter deformed differential structures on the quantum superplane (3). Concluding remarks follow in section 4.

2. Two-parameter quantum supergroup as Manin symmetry

Let the quantum matrix M, equation (1), be an element of a quantum supergroup such that we have

$$\begin{aligned} xa &= q_{11}xa & \theta a &= q_{21}a\theta \\ x\beta &= q_{12}\beta x & \theta\beta &= -q_{22}\beta\theta \\ x\gamma &= q_{13}\gamma x & \theta\gamma &= -q_{23}\gamma\theta \\ xd &= q_{14}dx & \theta d &= q_{24}d\theta \end{aligned} \tag{8}$$

where the q_{ij} are arbitrary complex parameters and let us assume that the quadratic relations (3) are transformed under M in (5) and its supertranspose ${}^{\mathrm{st}}M$ in (6), respectively, as

$$x'\theta' = \bar{q}\theta' x' \qquad \theta'^2 = 0 \tag{9}$$

and

$$x''\theta'' = \bar{\bar{q}}\theta'' x'' \qquad \theta''^2 = 0. \tag{10}$$

Then we obtain

(i) $M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \in GL_{p,q'}(1|1)$ for some non-zero p, q' with $pq' \neq -1$

(ii) $\bar{q} = \bar{\bar{q}}$

(iii)
$$\begin{aligned} q_{11} &= k & q_{21} &= q\bar{q}q'^{-1}p^{-1}k \\ q_{12} &= \bar{q}p^{-1}k & q_{22} &= q\bar{q}^2q'^{-1}p^{-2}k \\ q_{13} &= \bar{q}q'^{-1}k & q_{23} &= q\bar{q}^2q'^{-}2p^{-1}k \\ q_{14} &= \bar{q}^2q'^{-1}p^{-1}k & q_{24} &= q\bar{q}^3q'^{-2}p^{-2}k \end{aligned} \tag{11}$$

where k is an arbitrary complex number. Note that if one requires that $q_{ij} = 1$, then $\bar{q} = p = q' = q$ and $M \in GL_q(1|1)$.

In what follows, we will prove the above statement. The converse is trivial. Starting from equations (9) and (10), it follows that

$$\begin{aligned} &a\gamma = q_1\gamma a \qquad d\gamma = q_2\gamma d \\ &\gamma^2 = 0 \qquad \text{if } q_{13} \neq 0 \\ &qq_{14}ad - \bar{q}q_{21}da = q\bar{q}q_{12}\gamma\beta + q_{23}\beta\gamma \end{aligned} \tag{12}$$

and

$$\begin{aligned} &a\beta = q_3\beta a \qquad d\beta = q_4\beta d \\ &\beta^2 = 0 \qquad \text{if } q_{12} \neq 0 \\ &qq_{14}ad - \bar{\bar{q}}q_{21}da = -q\bar{\bar{q}}q_{13}\beta\gamma - q_{22}\gamma\beta \end{aligned} \tag{13}$$

where $q_1 = \bar{q}q_{11}q_{13}^{-1}$, $q_2 = qq_{14}q_{23}^{-1}$, $q_3 = \bar{\bar{q}}q_{11}q_{12}^{-1}$ and $q_4 = qq_{14}q_{22}^{-1}$.

If we require that the matrix M be an element of a quantum supergroup, then the above relations (12) and (13) must be consistent with the costructure of the Hopf algebra. The coproduct Δ and the antipode S are defined [13] by

$$\Delta\left(\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}\right) = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} a\otimes a + \beta\otimes\gamma & a\otimes\beta + \gamma\otimes d \\ \gamma\otimes a + d\otimes\gamma & \gamma\otimes\beta + d\otimes d \end{pmatrix} \tag{14}$$

and

$$S\left(\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}\right) = \begin{pmatrix} S(a) & S(\beta) \\ S(\gamma) & S(d) \end{pmatrix} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}^{-1}. \tag{15}$$

Note that the multiplication in the tensor product of the quantum supergroups is defined by $(a\otimes b)(c\otimes d) = (-1)^{\hat{b}\hat{c}}(ac\otimes bd)$ where, as usual, $\hat{b}$ denotes the parity of b (equal to 0 for bosonic generators and to 1 for fermionic ones). So, from the consistence $\Delta(\gamma^2) = 0$, $\Delta(a\gamma) = q_1\Delta(\gamma a)$ and $\Delta(d\gamma) = q_2\Delta(\gamma d)$, it follows that $q_1 = q_2 = q'$ and

$$ad - da = q'\gamma\beta + q'^{-1}\beta\gamma. \tag{16}$$

On the other hand, the consistence of $\beta^2 = 0$, $a\beta = q_3\beta a$ and $d\beta = q_4\beta d$ with Δ implies that $q_3 = q_4 = p$ and

$$ad - da = -p\beta\gamma - p^{-1}\gamma\beta \tag{17}$$

from which and (16) we deduce that

$$p\beta\gamma + q'\gamma\beta = 0 \tag{18}$$

unless $pq' = -1$. Hence, we obtain the two-parameter quantum deformation $GL_{p,q'}(1|1)$ of $GL(1|1)$, namely

$$\begin{aligned} &a\beta = p\beta a && d\beta = p\beta d \\ &a\gamma = q'\ ga && d\gamma = q'\gamma d \\ &p\beta\gamma + q'\gamma\beta = 0 && \beta^2 = \gamma^2 = 0 \\ &ad - da = (q'^{-1} - p)\beta\gamma && \end{aligned} \tag{19}$$

which proves that $M \in GL_{p,q'}(1|1)$ with $pq' \neq -1$. Next, the definition of the antipode S given by (15) implies the existence of the inverse matrix M^{-1}. So, if we set

$$M_{\mathrm{L}}^{-1} = \begin{pmatrix} \Delta_1^{-1} & 0 \\ 0 & \Delta_2^{-1} \end{pmatrix} \begin{pmatrix} d & m\beta \\ n\gamma & a \end{pmatrix} = \begin{pmatrix} \Delta_1^{-1}d & m\Delta_1^{-1}\beta \\ n\Delta_2^{-1}\gamma & \Delta_2^{-1}a \end{pmatrix} \tag{20}$$

such that $M_{\mathrm{L}}^{-1}M = 1$, we obtain $m = -p$, $n = -q'$, $\Delta_1 = da - p\beta\gamma$ and $\Delta_2 = ad - q'\gamma\beta$. On the other hand, if we take the right inverse M_{R}^{-1} of M as

$$M_{\mathrm{R}}^{-1} = \begin{pmatrix} sd & m'\beta \\ n'\gamma & s'a \end{pmatrix} \begin{pmatrix} \Delta_1^{-1} & 0 \\ 0 & \Delta_2^{-1} \end{pmatrix} = \begin{pmatrix} sd\Delta_1^{-1} & m'\beta\Delta_2^{-1} \\ n'\gamma\Delta_1^{-1} & s'a\Delta_2^{-1} \end{pmatrix} \tag{21}$$

it follows that $s = s' = 1$, $m' = -p^{-1}$, $n' = -q'^{-1}$, $\Delta_1 = ad - q'^{-1}\beta\gamma$ and $\Delta_2 = da - p^{-1}\gamma\beta$. The elements Δ_1 and Δ_2

$$\Delta_1 = da - p\beta\gamma = ad - q'^{-1}\beta\gamma \tag{22a}$$

$$\Delta_2 = ad - q'\gamma\beta = da - p^{-1}\gamma\beta \tag{22b}$$

are consistent with relations (17) and (18) and satisfy the following commutation relations:

$$\begin{aligned}
&a\Delta_1 - pq'\Delta_1 a = (1 - pq')\Delta_2 a \\
&p^{-1}\Delta_1\beta = p\beta\Delta_1 = p^{-1}\Delta_2\beta = p\beta\Delta_2 \\
&q'^{-1}\Delta_1\gamma = q'\gamma\Delta_1 = q'^{-1}\Delta_2\gamma = q'\gamma\Delta_2 \\
&d\Delta_1 = \Delta_1 d \\
&a\Delta_2 = \Delta_2 a \\
&d\Delta_2 - \Delta_2 d = (1 - pq')\Delta_1 d \\
&\Delta_1\Delta_2 + \Delta_2\Delta_1 = \Delta_1^2 + \Delta_2^2
\end{aligned} \tag{23}$$

from which it follows that

$$\begin{aligned}
&\Delta_1^{-1}a - pq'a\Delta_1^{-1} = (1 - pq')\Delta_2^{-1}a + (q'^{-1} + p)(p^{-2}q'^{-2} - 1)\beta\gamma\Delta_2^{-2}a \\
&p^{-1}\beta\Delta_1^{-1} = p\Delta_1^{-1}\beta = p^{-1}\beta\Delta_2^{-1} = p\Delta_2^{-1}\beta \\
&q'^{-1}\gamma\Delta_1^{-1} = q'\Delta_1^{-1}\gamma = q'^{-1}\gamma\Delta_2^{-1} = q'\Delta_2^{-1}\gamma \\
&\Delta_1^{-1}d = d\Delta_1^{-1} \\
&\Delta_2^{-1}a = a\Delta_2^{-1} \\
&\Delta_2^{-1}d - pq'd\Delta_2^{-1} = (1 - pq')\Delta_1^{-1}d - (p^{-2}q'^{-2} - 1)(q'^{-1} + p)\beta\gamma\Delta_1^{-2}d \\
&\Delta_1^{-1}\Delta_2^{-1} + \Delta_2^{-1}\Delta_1^{-1} = \Delta_1^{-2} + \Delta_2^{-2}
\end{aligned} \tag{24}$$

which are consistent with

$$M_{\mathrm{L}}^{-1} = \begin{pmatrix} \Delta_1^{-1}d & -p\Delta_1^{-1}\beta \\ -q'\Delta_2^{-1}\gamma & \Delta_2^{-1}a \end{pmatrix} = \begin{pmatrix} d\Delta_1^{-1} & -p^{-1}\beta\Delta_2^{-1} \\ -q'^{-1}\gamma\Delta_1^{-1} & a\Delta_2^{-1} \end{pmatrix} = M_{\mathrm{R}}^{-1}. \tag{25}$$

Now let us return to the third equations in (12) and (13). They should be identical to the relation (17). If $qq_{14} \neq \bar{q}q_{21}$, one writes the third equation in (12) as $qq_{14}ad - \bar{q}q_{21}da = r\beta\gamma$ where $r = q_{23} - q\bar{q}pq'^{-1}q_{12}$. In the case when $r = 0$, we have $ad = \epsilon da$ where $\epsilon = \bar{q}q_{21}q^{-1}q_{14}^{-1} \neq 1$. However, the consistence with the fact that $\Delta(ad) = \epsilon\Delta(da)$ leads us to $\epsilon = 1$, which yields a contradiction. When $r \neq 0$, there are two cases: $p \neq q'^{-1}$ and $p = q'^{-1}$. For the first case, according to the relation (17), we have $(qq_{14}(q'^{-1} - p) - r)ad = (\bar{q}q_{21}(q'^{-1} - p) - r)da$ which, in all possible cases, contradicts the fact that Δ_1 and Δ_2 are invertibles or that $\epsilon \neq 1$. In the second case when $p = q'^{-1}$, it follows that $ad = da = s\beta\gamma$ for some number s. However, the consistence $\Delta(ad) = s\Delta(\beta\gamma)$ implies $s = -p = q'^{-1}$, which contradicts the existence of Δ_1 and Δ_2

and the fact that $p = q'^{-1}$. So we conclude that $qq_{14} = \bar{q}q_{21}$. From a completely analogous discussion, we obtain $qq_{14} = \bar{\bar{q}}q_{21}$ from the third equation in (13). Hence we have

$$\bar{q} = \bar{\bar{q}}. \tag{26}$$

Finally, we find the following conditions on the q_{ij}:

$$\begin{aligned}
q' &= \bar{q}q_{11}q_{13}^{-1} = qq_{14}q_{23}^{-1} \\
p &= \bar{q}q_{11}q_{12}^{-1} = qq_{14}q_{22}^{-1} \\
qq_{14} &= \bar{q}q_{21} \\
q'^{-1} - p &= q^{-1}q'^{-1}pq_{14}^{-1}q_{22} - \bar{q}q_{14}^{-1}q_{13}.
\end{aligned} \tag{27}$$

Furthermore, if we multiply the two sides of the equality (17) by x and θ from the left and pull them from the right, we obtain two additional conditions:

$$\begin{aligned}
q_{11}q_{14} &= q_{12}q_{13} \\
q_{21}q_{24} &= q_{22}q_{23}
\end{aligned} \tag{28}$$

respectively. A straightforward calculation shows that we can express all of the q_{ij} in terms of one unknown $q_{11} = k$ which may be considered as a proportional constant. Explicitly, we find the relations (11). This completes the proof of the statement.

From relations (19), it is obvious that $M \in GL_{p,q'}(1|1)$ if and only if ${}^{\mathrm{st}}M \in GL_{q',p}(1|1)$. On the other hand, one can consider $GL_{p,q'}(1|1) = GL_{q',p}(1|1)$ in the sense that $GL_{p,q'}(1|1)$ and $GL_{q',p}(1|1)$ form the same $\mathbb{Z}_2$-graded free algebra generated by a, d, β, γ, Δ_1^{-1} and Δ_2^{-1} modulo the relations (19) and (24) and by the equations $(ad - p\beta\gamma)\Delta_1^{-1} - 1$, $\Delta_1^{-1}(ad - p\beta\gamma) - 1$, $(ad - q'\beta\gamma)\Delta_2^{-1} - 1$ and $\Delta_2^{-1}(ad - q'\beta\gamma) - 1$. Thus the Manin approach considering the quantum supergroup as supersymmetry group is recovered under the q-supercommutation relations (8) between the quantum supergroup generators and the coordinates of the quantum superplane, where the q_{ij} are given by (11). We see that, if we take $q = \bar{q} = q' = p$, the parameter k remains arbitrary and the choice $k = 1$ appears as a particular case. This means that also in the one-parameter case, one can take the generators of the quantum supergroup and the coordinates of the quantum superplane to be non-supercommutative. The assumption of supercommutativity is very special.

3. Quantum superplanes

This section is devoted to the discussion of several interesting choices of the q_{ij}. The different possibilities of choosing the q_{ij} imply the possibility of constructing many quantum superplanes associated with a given quantum supergroup. This formulation allows us to make a correspondence between the one and two-parameter deformed differential structures on the quantum superplane.

3.1. Case 1: $\bar{q} = q$

This case is the standard one of dealing with the quantum superplane. Then, taking $q_{11} = k = 1$ for simplicity, we have

$$\begin{aligned}
q_{11} &= 1 & q_{21} &= q_{14} = q^2q'^{-1}p^{-1} \\
q_{12} &= qp^{-1} & q_{22} &= q^3q'^{-1}p^{-2} \\
q_{13} &= qq'^{-1} & q_{23} &= q^3q'^{-2}p^{-1} \\
q_{14} &= q^2q'^{-1}p^{-1} & q_{24} &= q^4q'^{-2}p^{-2}.
\end{aligned} \tag{29}$$

Now, let us introduce the graded exterior differential d as in [7, 14, 15] except for the following relations:

$$
\begin{aligned}
&(dx)a = q_{11}a\,dx && (d\theta)a = q_{21}a\,d\theta \\
&(dx)\beta = -q_{12}\beta\,dx && (d\theta)\beta = q_{22}\beta\,d\theta \\
&(dx)\gamma = -q_{13}\gamma\,dx && (d\theta)\gamma = q_{23}\gamma\,d\theta \\
&(dx)d = q_{12}d\,dx && (d\theta)d = q_{24}d\,d\theta
\end{aligned}
\tag{30}
$$

where the q_{ij} are given by (29), and suppose that the quadratic relation $dx\,d\theta = (1/p)d\theta\,dx$ of the exterior quantum superplane is preserved under the following transformations:

$$
M' = \begin{pmatrix} a & -\beta \\ -\gamma & d \end{pmatrix} \qquad \text{and} \qquad {}^{\mathrm{st}}M' = \begin{pmatrix} a & \gamma \\ -\beta & d \end{pmatrix}
\tag{31}
$$

respectively. Then we find $q = q'$ and consequently the relations (29) reduce to

$$
\begin{aligned}
&q_{11} = q_{13} = 1 \\
&q_{12} = q_{14} = q_{21} = q_{23} = qp^{-1} \\
&q_{22} = q_{24} = q^2 p^{-2}.
\end{aligned}
\tag{32}
$$

With this formulation, we see that the quadratic relation between the differentials on the quantum superplane is invariant not only under transformation M' but also under its supertranspose. Note that M' is an element of $GL_{p,q'}(1|1)$ as well as ${}^{\mathrm{st}}M'$. In the following we shall suppress the prime sign when we deal with transformations (31) of differentials. This will not leads to ambiguity.

Now let us look at differential structure on the quantum superplane. This was investigated by several authors [7, 14–16] both in the one or two-parameter deformation cases. The latter investigations are based on the introduction of an exterior differential operator d which satisfies the linearity, nilpotency ($d^2 = 0$) and the graded Leibnitz rule. In the one-parameter case [16], one can choose

$$
\begin{aligned}
&dx\,d\theta = \frac{1}{q}d\theta\,dx \\
&x\,dx = q^2(dx)x \\
&x\,d\theta = q(d\theta)x + (q^2 - 1)(dx)\theta \\
&\theta\,dx = -q(dx)\theta \\
&\theta\,d\theta = (d\theta)\theta.
\end{aligned}
\tag{33}
$$

It is easy to see that the relations (33) are invariant under the transformation M and its supertranspose. The two-parameter case may be obtained using the same method [7] [15]. One obtains the following scheme:

$$
\begin{aligned}
&dx\,d\theta = \frac{1}{p}d\theta\,dx \\
&x\,dx = pq(dx)\,dx \\
&x\,d\theta = q(d\theta)x + (pq - 1)(dx)\theta \\
&\theta\,dx = -p(dx)\theta \\
&\theta\,d\theta = (d\theta)\theta.
\end{aligned}
\tag{34}
$$

In this case, one can easily check that the relations (34) are not invariant under transformation ${}^{\mathrm{st}}M$ if one assumes that the generators a, β, γ and d of the quantum supergroup supercommute with the coordinates x, θ of the quantum superplane. However, if we choose the parameters q_{ij} as in (32), then it is straightforward, though tedious, to verify that the relations (34) are invariant not only under M but also under ${}^{\mathrm{st}}M$. Moreover, we have $dx'\,dx' = 0$ and $dx''\,dx'' = 0$.

3.2. Case 2: $p = q'$

This case corresponds to the one parameter quantum supergroup $GL_{q'}(1|1)$. The parameters (q_{ij}) in (11) become

$$\begin{aligned} &q_{11} = 1 && q_{21} = q\bar{q}q'^{-2} \\ &q_{12} = q_{13} = \bar{q}q'^{-1} && q_{22} = q_{23} = q\bar{q}^2q'^{-3} \\ &q_{14} = \bar{q}^2q'^{-2} && q_{24} = q\bar{q}^3q'^{-4}. \end{aligned} \tag{35}$$

The quantum superplane such that $x\theta = q\theta x$ corresponding to the values (35) of the q_{ij} is transformed into $x'\theta' = \bar{q}\theta'x'$ and $x''\theta'' = \bar{q}\theta''x''$ under M and its supertranspose, respectively. Now, if we introduce the differentials dx and $d\theta$ such that $dx\,d\theta = (1/p')d\theta\,dx$ and transformed into $dx'\,d\theta' = (1/\bar{p}')d\theta'\,dx'$ and $dx''\,d\theta'' = (1/\bar{p}')d\theta''\,dx''$ under M and its supertranspose, respectively, then we must take $p' = q'^2q^{-1}$ and $\bar{p}' = q'^2\bar{q}^{-1}$ when the q_{ij} are given by (35). In this case the scheme (34) becomes

$$\begin{aligned} &dx\,d\theta = \frac{q}{q'^2}d\theta\,dx \\ &x\,dx = q'^2(dx)x \\ &x\,d\theta = q(d\theta)x + (q'^2 - 1)(dx)\theta \\ &\theta\,dx = -\frac{q'^2}{q}(dx)\theta \\ &\theta\,d\theta = (d\theta)\theta. \end{aligned} \tag{36}$$

So, under transformation M we have

$$\begin{aligned} &dx'\,d\theta' = \frac{\bar{q}}{q'^2}d\theta'\,dx' \\ &x'\,dx' = q'^2(dx')x' \\ &x'\,d\theta' = \bar{q}(d\theta')x' + (q'^2 - 1)(dx')\theta' \\ &\theta'\,dx' = -\frac{q'^2}{\bar{q}}(dx')\theta' \\ &\theta'\,d\theta' = (d\theta')\theta' \end{aligned} \tag{37}$$

and under the transformation ${}^{\mathrm{st}}M$ we have

$$\begin{aligned}
&dx''\,d\theta'' = \frac{\bar{q}}{q'^2} d\theta''\,dx'' \\
&x''\,dx'' = q'^2 (dx'')x'' \\
&x''d\theta'' = \bar{q}(d\theta'')x'' + (q'^2 - 1)(dx'')\theta'' \\
&\theta''\,dx'' = -\frac{q'^2}{\bar{q}}(dx'')\theta'' \\
&\theta''\,d\theta'' = (d\theta'')\theta''.
\end{aligned} \tag{38}$$

Thus, we obtain the two-parameter differential scheme on the quantum superplane which is covariant under the one-parameter quantum supergroup $GL_{q'}(1|1)$. This formulation allows us to make correspondence between the one and two-parameter deformed differential schemes as follows. First, if we take $q = q'$, then the parameters (q_{ij}), equations (35), reduce to the following:

$$\begin{aligned}
&q_{11} = 1 \\
&q_{12} = q_{13} = q_{21} = \bar{q}q^{-1} \\
&q_{14} = q_{22} = \bar{q}^2 q^{-2} \\
&q_{24} = \bar{q}^3 q^{-3}
\end{aligned} \tag{39}$$

and the scheme (36) reduces to the one in (33) depending only on one deformation parameter. But under transformation M and its supertranspose, we obtain the differential schemes (37) and (38), respectively. Hence, we obtain in this way the two-parameter differential scheme from the one depending only on one deformation parameter. Second, if we take $\bar{q} = q'$, then the expressions (35) of (q_{ij})'s become

$$q_{1i} = 1 \qquad q_{2i} = qq'^{-1} \tag{40}$$

where $i = 1, \ldots, 4$ and the schemes (37) and (38) reduce to the one-parameter deformed differential schemes obtained from the two-parameter one in (36) under transformation M and its supertranspose, respectively. Note that these differential constructions may also be performed in the case of the quantum plane, although this was not done in [8]. Furthermore, if we take $q = 1$, then

$$q_{1i} = 1 \qquad q_{2i} = q'^{-1} \tag{41}$$

for $i = 1, \ldots, 4$. This corresponds to the classical superplane $x\theta = \theta x$ and $\theta^2 = 0$ which is transformed under the action of quantum supergroup $GL_{q'}(1|1)$ to the quantum one, $x'\theta' = q'\theta' x'$ and $\theta'^2 = 0$. But these new non-supercommutative coordinates do not obey (8). Finally, in the case where $q = \bar{q} = 1$ we have

$$\begin{aligned}
&q_{11} = 1 \\
&q_{12} = q_{13} = q'^{-1} \\
&q_{14} = q_{21} = q'^{-2} \\
&q_{22} = q_{23} = q'^{-3} \\
&q_{24} = q'^{-4}.
\end{aligned} \tag{42}$$

This case also seems to be of interest, since the quantum superplane looks like an ordinary superplane in the sense that it is generated by supercommutative coordinates. The quantum superplane corresponding to $GL_{q'}(1|1)$ such that $q = \bar{q} = q'$ is the original one [5]. Indeed in this case all of the q_{ij} are equal to 1.

4. Concluding remarks

The usual Manin approach that considers the quantum group as the symmetry group of the quantum plane is based on the remarkable assumption that generators of the quantum group commute with the coordinates of the quantum plane. The condition that the quadratic relation $xy = qyx$ is preserved under the action of the quantum matrix M and its transpose gives the commutation relations of the quantum group $GL_q(2)$. One can never obtain the two-parameter quantum group. In this paper, we extended the above approach to the supercase. It is also valid only in the one-parameter deformation case. Next, we have relaxed the basic assumption of supercommutativity between coordinates and generators and investigated its consequences. We are led in a natural way to the two-parameter deformation of the supergroup $GL(1|1)$ and its corresponding quantum superplanes though we do not put any restriction on the number of parameters at the outset. With this formulation, the Manin's approach considering that quantum supergroups are supersymmetry groups of quantum superplanes is still preserved and the different choices of the q_{ij} show that there are many quantum superplanes for a given quantum supergroup $GL_{p,q}(1|1)$. There are some interesting quantum superplanes such as the original one in the literature and the classical superplane. When we considered the differential structure on the quantum superplane, this formulation allows us to obtain the one-parameter differential scheme from the two-parameter one and conversely according to special choices of the q_{ij}.

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